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A SYNOPSIS OF KNOWN DISTANCE-REGULAR GRAPHS WITH LARGE DIAMETERS

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A synopsis of known distance-regular graphs with large diameters

by

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ABSTRACT

The known series of distance-regular graphs in which the diameter occurs as a parameter, are reviewed. These notes are inspired by and more or less grown out of a lecture by Professor Eiichi Bannai given at Oberwolfach in May, 1980.

KEY WORDS & PHRASES : *distance regular graphs, near n-gons*

0. NOTATION, TERMINOLOGY AND INTRODUCTORY REMARKS

0.1 DEFINITIONS. All graphs in these notes are finite and without loops or multiple edges. Let Γ be a graph. By $\gamma \in \Gamma$ we mean that γ is a vertex of Γ and by $\gamma\Gamma\delta$ or $\gamma \in \Gamma(\delta)$ for $\gamma,\delta \in \Gamma$ we mean that $\{\gamma,\delta\}$ is an edge of Γ. The cardinality $|\Gamma|$ of the vertex set of Γ is denoted by v. For $\gamma,\delta \in \Gamma$ the usual distance between them is $d(\gamma,\delta)$. If $\delta \in \Gamma$ and $i \in \mathbb{Z}_{\geq 0}$ then $\Gamma_i(\delta)$ stands for the set of vertices γ with $d(\gamma,\delta) = i$ and $\Gamma_{\leq i}(d)$ for $\cup_{0\leq j\leq i} \Gamma_j(\delta)$. Moreover, $\Gamma(\delta) = \Gamma_1(\delta)$ in accordance with the definition of $\gamma \in \Gamma(\delta)$. The diameter of Γ is denoted by d. To avoid trivialities we assume $d > 1$. For $\gamma \in \Gamma$, $\delta \in \Gamma_j(\gamma)$, we write $c(\gamma,\delta) = |\Gamma_{j-1}(\gamma) \cap \Gamma_1(\delta)|$, $b(\gamma,\delta) = |\Gamma_{j+1}(\gamma) \cap \Gamma_1(\delta)|$, $a(\gamma,\delta) = |\Gamma_j(\gamma) \cap \Gamma_1(\delta)|$ and $k_j(\delta) = |\Gamma_j(\delta)|$. If $c(\gamma,\delta)$ is independent of the choice of $\gamma,\delta \in \Gamma$ with $d(\gamma,\delta) = j$, then we write c_j instead of $c(\gamma,\delta)$ and say that c_j *exists*. Similarly with b_j, a_j and k_j for $b(\gamma,\delta)$, $a(\gamma,\delta)$ and $k_j(\delta)$ respectively. Γ is called *distance-regular* if it is connected and a_j, b_j, c_j, k_j exist for all j.

Let Γ be a distance-regular graph. Then the ordered sequence $\{b_0,b_1,\ldots,b_{d-1}; c_1,c_2,\ldots,c_d\}$ is called the *intersection array* of Γ. Clearly, $k_1 = b_0$ and $c_1 = 1$. For $h,i,j \in \mathbb{Z}_{\geq 0}$, choose $\gamma,\delta \in \Gamma$ with $\gamma \in \Gamma_j(\delta)$ and write $s_{h,i,j} = |\Gamma_h(\gamma) \cap \Gamma_i(\delta)|$. Then $s_{h,i,j}$ is independent of the chosen γ, δ with $\gamma \in \Gamma_j(\delta)$ and can be determined from the intersection array. More precisely, denote by B_h the $(d+1)\times(d+1)$-matrix whose (i,j)-coefficient $(B_h)_{i,j}$ is $s_{h,i-1,j-1}$, then the following holds.

0.2 PROPOSITION. *Let Γ be distance regular with intersection array* $\{b_0,b_1,\ldots,b_{d-1}; c_1,c_2,\ldots,c_d\}$. *Then*

(i) $k_j = b_0b_1\ldots b_{j-1}/c_1c_2\ldots c_j$, $(1 \leq j \leq d)$;

(ii) $a_j = k - b_j - c_j$;

(iii) B_1 *is tridiagonal and satisfies* $B_1B_i = b_{i-1}B_{i-1} + a_iB_i + c_{i+1}B_{i+1}$.

(iv) B_1 *has* d+1 *distinct eigenvalues* λ_i $(0 \leq i \leq d)$ *corresponding to left eigenvectors* u_i *and right eigenvectors* v_i *satisfying* $(u_i)_0 = (v_i)_0 = 1$.

(v) *Let* λ_i, u_i, v_i *be as in* (iv). *Then* $(v_i)_j = k_j(u_i)_j$ *for all* i, j $(0 \leq i,j \leq d)$, *and the multiplicities* $m(\lambda_i)$ *of* λ_i *as eigenvalues of the adjacency matrix of* Γ *are given by*

$$m(\lambda_i) = v/(u_i, v_i), \quad (0 \le i \le d),$$

where (u_i, v_i) *is the standard inner product of* u_i *and* v_i.

For more details, the reader is referred to BIGGS [5], or to MacWILLIAMS and SLOANE [21]. In GARDINER [19], some inequalities relating a_j, b_j, c_j are derived.

0.3 DEFINITIONS. For $\gamma \in \Gamma$, an alternative way of denoting $\Gamma_{\le 1}(\gamma)$ is $\gamma^\perp$. Moreover, if X is a set of vertices, $X^\perp$ stands for $\bigcap_{\gamma \in X} \gamma^\perp$. If Γ is a graph for which a_1 exists, then Γ is the collinearity graph of a linear incidence system whose points are the vertices of Γ and whose lines are the subsets $\{\gamma,\delta\}^{\perp\perp}$ for $\gamma,\delta \in \Gamma$ with $\gamma \in \Gamma(\delta)$. These lines will be called *singular lines*, cf. [8]. If the automorphism group Aut(Γ) of Γ is transitive on these pairs γ,δ , then all lines have the same size. In general, $1+s(\gamma,\delta)$ will denote the line size. Often, when γ, δ are clear from the context or when this number is independent of the chosen $\gamma,\delta \in \Gamma$ with $\gamma \in \Gamma(\delta)$, we shall abbreviate to $1+s$ and say that s exists. Moreover, $h(\Gamma)$, or just h when Γ is clear from the context, will denote the set of all lines $\{\gamma,\delta\}^{\perp\perp}$ for $\gamma \in \Gamma(\delta)$.

If H is a subgroup of the automorphism group Aut(Γ) of Γ such that for each i $(0 \le i \le d)$ the group is transitive on the set $\{(\gamma,\delta) \mid \gamma \in \Gamma_i(\delta)\}$ then H is called *distance-transitive* (on Γ). A graph Γ is called *distance-transitive* if it admits a distance-transitive group of automorphisms. Clearly, such a graph is distance-regular.

0.4 REMARKS. One of the major goals (BANNAI [2]) in the theory of distance-regular graphs is (or should be) to find all distance regular graphs of sufficiently large diameter, or even of diameter > 12. Let K be the set of all known such graphs. If K is all there is, a natural first step in the proof of the desired theorems would be to restrict all feasible intersection arrays to those arising from K, while the natural second stage would consist of a uniqueness proof for the examples in K with a given intersection array. Most likely, completely different techniques are needed for these two steps.

So far, only very partial results are known. Representation techniques applicable in the first stage are excellently surveyed in OTT [22]. BANNAI and ITO's result [4] may also be considered as belonging to this stage.

Some theorems that fall under the second stage are quoted in Section 2 under 'Additional properties' in the relevant subsections. Here, we restrict to the observation that once the existence of singular lines of the right cardinality (s+1) is established, the uniqueness problem seems tractable. Since knowledge of K is necessary to any work in this area, it is hoped that these notes may constitute relevant background material for anyone interested in distance-regular graphs.

1. THE KNOWN GRAPHS IN OVERVIEW

The known series of distance-regular graphs (parametrized by the diameter) are listed in Table I. For the sake of presentation, we formulate this fact as a theorem.

1.1 THEOREM. *Let* Γ *be one of the graphs of Table* I, *whose construction is given at the beginning of one of the sub-sections of Section* 2. *Then* Γ *is distance-regular and* v, b_j, c_j, k_j, s, d *are as specified in the table. Finally* Aut Γ *has a subgroup as given in the table.*

Indications of the proof as well as some additional properties will be given in Section 2. The part of the theorem that applies to graph Γ is referred to as Theorem (Γ).

In these notes we shall hardly pay attention to the matrix techniques, that involve study of the eigenvalues of the adjacency matrix and their multiplicities and of the so-called Q-matrices among many other aspects. For each of these graphs, the corresponding eigenvalues and their multiplicities only depend on their intersection arrays. References to information of this kind are given in 3.1.

1.2 REMARK. There are more series than those described in the theorem. For instance, there are distance-regular graphs obtained from other distance-regular graphs by one of the two processes *folding* and *halving*.

TABLE I

Intersection arrays of known distance-regular graph Γ of diameter d ($d \geq 3$).

n, r are natural numbers, q is a prime power and $\begin{bmatrix} n \\ r \end{bmatrix}_q$ denotes $\prod_{i=0}^{r-1} (q^n - q^i)/(q^r - q^i)$

Notation for Γ	Defined in	Name	b_j $(-1<j<d)$	c_j $(0<j<d+1)$	line length $s+1$	a subgroup G of $\mathrm{Aut}(\Gamma)$
J	2.1.1	Johnson (d,n)	$(d-j)(n-d-j)$	j^2	2	$\mathrm{Sym}(n)$
O	2.1.5	Odd graph $(d+1)$	$d+1, d, d, d-1, d-1, \ldots$	$1,1,2,2,3,3,\ldots$	2	$\mathrm{Sym}(n)$
Ja	2.2.1	q-analog of Johnson (d,n)	$q^{2j+1} \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \begin{bmatrix} n-d-j \\ 1 \end{bmatrix}_q$	$\begin{bmatrix} j \\ 1 \end{bmatrix}_q^2$	$q+1$	$P\Gamma L(n,q)$
E with e= 0	2.3.1	$Sp(2d,q)=C_d(q)$	$q^{j+e+1} \begin{bmatrix} d-j \\ 1 \end{bmatrix}_q$	$\begin{bmatrix} j \\ 1 \end{bmatrix}_q$	$q^{e+1}+1$	$P\Gamma Sp(2d,q)$
0		$\Omega(2d+1,q)=B_d(q)$				$P\Gamma O(2d+1,q)$
-1		$\Omega^+(2d,q)=D_d(q)$				$P\Gamma O^+(2d,q)$
1		$\Omega^-(2d+2,q)={}^2D_{d+1}(q)$				$P\Gamma O^-(2d,q)$
$\frac{1}{2}$		$U(2d+1,r)={}^2A_{2d}(r)$ for $r^2=q$				$P\Gamma U(2d+1,r)$
$-\frac{1}{2}$		$U(2d,r)={}^2A_{2d-1}(r)$ for $r^2=q$				$P\Gamma U(2d,r)$
H	2.4.1	Hamming (d,r)	$(d-j)(r-1)$	j	r	$\mathrm{Sym}(r) \wr \mathrm{Sym}(d)$
Ha	2.5.1	q-analog of Hamming $(d,n+d)$	$q^{2j}(q-1)\begin{bmatrix} d-j \\ 1 \end{bmatrix}_q \begin{bmatrix} n+d-j \\ 1 \end{bmatrix}_q$	$q^{j-1}\begin{bmatrix} j \\ 1 \end{bmatrix}_q$	q	$\mathbb{F}_q^{(n+d)d} \cdot GL(d,q) \times \Gamma L(d+n,q)/\mathbb{F}_q^*)$
Alt	2.6.1	Alternating forms on n-dim space for $n \in \{2d, 2d+1\}$	$q^{4j}(q-1)\begin{bmatrix} n-2j \\ 2 \end{bmatrix}_q$	$q^{2j-2}\begin{bmatrix} j \\ 1 \end{bmatrix}_{q^2}$	q	$\mathbb{F}_q^{n(n-1)/2} \cdot GL(n,q)/\{\pm 1\}$
Her	2.7.1	Hermitian forms on d-dim space, $q=r^2$	$q^j(r-1)\begin{bmatrix} d-j \\ 1 \end{bmatrix}_q$	$r^{j-1}(r^j-(-1)^j)/(r+1)$	r	$\mathbb{F}_r^{d^2} \cdot GL(d,q)/K$ where $K=\{x \in \mathbb{F}_q \mid x^{r+1}=1\}$
Q	2.8.1	Quadratic forms on n-dim space for $n \in \{2d-1, 2d\}$	$q^{4j}(q-1)\begin{bmatrix} n-2j+1 \\ 2 \end{bmatrix}_q$	$q^{2j-2}\begin{bmatrix} j \\ 1 \end{bmatrix}_{q^2}$	$2,q$	$\mathbb{F}_q^{n(n+1)/2} \cdot GL(n,q)/\{\pm 1\}$
I	2.9.1	$2d$-gon	$2,1,\ldots,1$	$1,\ldots,1,2$	2	D_{2d}
		$(2d+1)$-gon	$2,1,\ldots,1$	$1,\ldots,1,1$	2	D_{2d+1}

Suppose Γ is a distance-regular graph of diameter d in which being *opposite* (i.e. of distance d) is an equivalence relation. Then $\hat{\Gamma}$, the set of equivalence classes with adjacency $\hat{x}\hat{\Gamma}\hat{y}$ for $\hat{x},\hat{y} \in \hat{\Gamma}$ defined by $x\Gamma y$ for some $x \in \hat{x}$, $y \in \hat{y}$, is a distance-regular graph, called the *folded graph* of Γ. Folding may be applied to the Hamming graph H of diameter d on two points, or to the Johnson graph J of diameter d on 2d points or to the 2d-gon.

Suppose Γ is a bipartite distance-regular graph and Γ' is a member of a partitioning of Γ into two cocliques. Then adjacency $x\Gamma'y$ for $x,y \in \Gamma'$ defined by $x \in \Gamma_2(y)$ turns Γ' into a distance-regular graph, called the *halved graph* of Γ. Halving may be applied to the graph $\Omega^+(2d,q)$. (Since the parameters of folded and halved graphs are easily derived from those of the original graph, we shall not discuss them.)

But this is not all. EGAWA [17] has shown that there are precisely $[d/2]$ isomorphism classes of distance-regular graphs distinct from H(d,4) but with the same intersection arrays (cf. 2.4.3).

1.3. <u>REMARK</u>. According to LEONARD [20], only 6 parameters are necessary to describe the intersection array of a distance-regular graph if the corresponding association scheme is 'Q-polynomial'. As all the examples of the table satisfy this condition, there are more succinct ways to characterize the parameters of these graphs. However, we prefer the intersection arrays in order to be able to restrict to the geometrical aspects of these graphs.

2. THE KNOWN GRAPHS IN MORE DETAIL

2.1. The Johnson Graphs and the Odd Graphs

2.1.1. <u>DEFINITION</u>. Take X to be a finite set of cardinality $n \geq 2d$. The *Johnson graph* J (*of diameter* d *on* X) has vertex set $\binom{X}{d}$, the collection of d-subsets of X. Two points x,y of J are *adjacent* whenever $x \cap y$ has cardinality d-1.

2.1.2. PROOF OF THEOREM. (J) is straightforward. Note that $x \in J_i(y)$ iff $x \cap y$ has cardinality $d-i$. □

ADDITIONAL PROPERTY.

$\mathrm{Aut}(J) \cong \mathrm{Sym}(X)$, *the symmetric group on* X, *unless* $n = 2d$;

$\mathrm{Aut}(J) \cong 2.\mathrm{Sym}(X)$ *if* $n = 2d$.

PROOF. For $d = 1$ there is nothing to prove. Suppose $n > 2d > 2$ and use induction on d. Form a new graph Δ on the $\binom{n}{d-1}$ cliques of size $n-d+1$, in which two distinct such cliques α, β are adjacent whenever $\alpha \cap \beta$ is a singleton. Then Δ is isomorphic to the Johnson graph of diameter $d-1$ on X so by induction, $\mathrm{Aut}(\Delta) \cong \mathrm{Sym}(n)$. On the other hand, if an automorphism of J stabilizes all $(n-d+1)$-cliques of J, it fixes all vertices of J, for the singleton on each vertex of J occurs as the intersection of a pair of $(n-d+1)$-cliques. It follows that the natural morphism $\mathrm{Aut}(J) \to \mathrm{Aut}(\Delta)$ is injective. Since $\mathrm{Sym}(X) \le \mathrm{Aut}(J)$, comparison of orders leads to the desired isomorphism between $\mathrm{Aut}(J)$ and $\mathrm{Sym}(X)$.

In the case where $n = 2d$, the graph Δ of above has twice as many vertices. They fall into two components each of which is isomorphic to the Johnson graph of diameter $d-1$ on X. This implies $\mathrm{Aut}(J) = C_2 \times \mathrm{Sym}(X)$ as wanted. □

2.1.3. ADDITIONAL PROPERTY. Let Δ be a connected graph on $\binom{n}{d}$ vertices, regular of degree $d(n-d)$, such that for any $\gamma, \delta \in \Delta$ we have $a(\gamma,\delta) = n-2$ if $\gamma \in \Delta(\delta)$ and $|\{\gamma,\delta\}^{\perp}| \le 4$ if $\gamma \notin \Delta_{\le 1}(\delta)$. If $n > 2d(d-1)+4$, then $\Delta \cong J$.

PROOF. See DOWLING [16]. □

Brouwer has announced a relaxation of the lower bound for n in terms of d. MOON [30] has recently proved uniqueness of distance-regular graphs whose intersection array is that of a Johnson graph for $n > \frac{1}{3}(14d+10)$.

2.1.4. Though the following theorem is not quite a characterization by parameters, it is of sufficient interest to be mentioned here.

<u>THEOREM</u>. (SPRAGUE [26]). *Let Γ be a connected graph in which a collection $\mathfrak{m}$ of maximal cliques exists such that the following axioms hold:*

(i) *Each edge $\{\gamma,\delta\}$ of Γ is in a unique member of $\mathfrak{m}$;*

(ii) *If $L_1, L_2, M_1, M_2 \in \mathfrak{m}$ and $\gamma \in \Gamma$ such that $L_1 \cap L_2 = \{\gamma\}$ and $\gamma \notin L_i \cap M_j \neq \emptyset$ for all $i,j \in \{1,2\}$ then $M_1 \cap M_2 \neq \emptyset$;*

(iii) *If $\{\gamma,\delta\}$ is an edge of Γ and $\zeta,\eta \in \{\gamma,\delta\}^{\perp}$ are distinct points not in the member of $\mathfrak{m}$ containing $\{\gamma,\delta\}$, then $\zeta \in \Gamma(\eta)$;*

(iv) *If $\gamma \in \Gamma$ and $M \in \mathfrak{m}$, then either $\gamma \in M$ or $|\gamma^{\perp} \cap M| \in \{0,2\}$.*

Then Γ is isomorphic to a Johnson graph.

2.1.5. <u>DEFINITION</u>. The *Odd Graph* O of diameter d on X, where X has cardinality $n = 2d+1$ is defined on the points of the Johnson graph J on X, with adjacency given by $\gamma O d \iff \gamma \in J_d(\delta)$ for any $\gamma,\delta \in O$.

2.1.6. <u>PROOF OF THEOREM</u> (O): straightforward. Note that $\gamma \in O_{2j}(\delta)$ iff $\gamma \in J_j(\delta)$ for $j \in \{0,1,\ldots,\lfloor d/2 \rfloor\}$ and that $\gamma \in O_{2j+1}(\delta)$ iff $\gamma \in J_{d-j}(\delta)$ for $j \in \{0,1,\ldots,\lfloor d-1/2 \rfloor\}$. The maximal cliques of O are edges (and O has girth 6 if $d \geq 3$), so that $\{\gamma,\delta\}^{\perp\perp} = \{\gamma,\delta\}$ for any $\gamma \in O$ and $\delta \in O(\gamma)$, proving $s = 1$. $\square$

2.1.7. <u>ADDITIONAL PROPERTY</u>. The algebra generated by the adjacency matrix of O coincides with that of J, and $\mathrm{Aut}(O) = \mathrm{Aut}(J) \cong \mathrm{Sym}(X)$.

2.2. <u>The q-Analog of the Johnson Graphs</u>

2.2.1. <u>DEFINITION</u>. Set $V = \mathbb{F}_q^n$, where $n \geq 2d$, and q is a prime power. The *q-analog of the Johnson graph*, denoted Ja, *on* V *of diameter* d has vertex set $\left[{V \atop d}\right]$, the collection of linear subspaces of V of dimension d. Two points X, Y of Ja are *adjacent* whenever $\dim X \cap Y = d-1$. A linear subspace of V of dimension m is called an *m-space* (of V). Write $\left[{n \atop m}\right]$ for the number of m-spaces of V.

2.2.2. <u>LEMMA</u>

(i) $\left[{n \atop m}\right] = \prod_{i=0}^{m-1} (q^n - q^i)/(q^m - q^i)$

(ii) If X is a j-space of V, then $\#\{Y \subseteq V \mid Y \text{ is an } i\text{-space of } V \text{ and } X \cap Y = 0\} = q^{ij}\left[{n-j \atop i}\right]$.

(iii) If X is a j-space of V, then

$$\#\{Y \subseteq V \mid Y \text{ is an } i\text{-space and } X \cap Y \text{ is an } m\text{-space}\}$$

$$= q^{(i-m)(j-m)}\left[{n-j \atop i-m}\right]\left[{j \atop m}\right].$$

PROOF.

(i) There are $\prod_{i=0}^{m-1}(q^n-q^i)$ ordered m-tuples of linearly independent vectors in V, and $\prod_{i=0}^{m-1}(q^m-q^i)$ ordered bases of any m-space.

(ii) Given an ordered i-tuple of linearly independent vectors in V/X there are $q^{ij} = \#\mathrm{Hom}_{\mathbb{F}_q}(V/X,X)$ ways to lift this i-tuple to an i-tuple of linearly independent vectors in V.

(iii) Given an m-space Z of X, we have

$$\#\{Y \subseteq V \mid Y \text{ is an } i\text{-space, } Y \cap X = Z\}$$

$$= \#\{Y \subset V/Z \mid Y \text{ is an } (i-m)\text{-space, } (X/Z) \cap Y = 0\},$$

while the latter number is $q^{(i-m)(j-m)}\left[{n-j \atop i-m}\right]$ by (ii). In view of (i), there are $\left[{j \atop m}\right]$ such subspaces Z. □

2.2.3. PROOF OF THEOREM. (Ja). Clearly, $X,Y \in Ja$ have distance j if and only if $\dim(X \cap Y) = d-j$. Note that $P\Gamma L(n,q)$ is a subgroup of $\mathrm{Aut}(Ja)$. As $\Gamma L(n,q)$ is transitive on ordered bases, it follows readily that Γ is distance-transitive.

In order to obtain $b(j)$ and $c(j)$, fix $X,Y \in Ja$ with $X \in Ja_j(Y)$. Note that $Z \in Ja$ is in $Ja_{j-1}(X) \cap Ja(Y)$ iff $X \cap Y \subseteq Z \subseteq X+Y$, $\dim X \cap Z = d-j+1$ and $\dim Y \cap Z = d-1$. Calculating in $X+Y/X \cap Y$, we see that the number of such Z, which is $c(X,Y)$ by definition, equals the product of the number of 1-spaces in $X/X \cap Y$ and the number of $(j-1)$-spaces of $Y/X \cap Y$. This yields that $c_j = c(X,Y) = \left[{j \atop 1}\right]\left[{j \atop j-1}\right]$ as wanted. By (2.2.2(iii)), $k_j = qj^2\left[{n-d \atop j}\right]\left[{d \atop j}\right]$. In view of Proposition 0.2, this determines b_j.

Let us now determine the cardinality $s+1$ of singular lines. Suppose $X \in Ja$ and $Y \in Ja(X)$. Then

$$\{X,Y\}^{\perp} = \{Z \in Ja \mid Z \subseteq X+Y\} \cup \{Z \in Ja \mid Z \cap (X+Y) = X \cap Y\}.$$

It follows that

$$\{X,Y\}^{\perp\perp} = \{U \in Ja \mid U \subseteq X+Y \text{ and } U \cap X,\ U \cap Y \supseteq X \cap Y\},$$

is of cardinality $q+1$, proving $s = q$. □

2.2.4. <u>ADDITIONAL PROPERTY</u>. If $n > 2d$, then $\mathrm{Aut}(Ja) \cong P\Gamma L(n,q)$. If $n = 2d$, then $\mathrm{Aut}(Ja) \cong \mathrm{Aut}\, P\Gamma L(n,q)$, the latter group being an extension of $P\Gamma L(n,q)$ of degree 2.

<u>PROOF</u>. In general, Ja has maximal cliques of sizes $\left[{n-d+1 \atop 1}\right]$ and $\left[{d+1 \atop 1}\right]$ corresponding to the d-spaces containing a given (d-1)-space and the d-spaces contained in a (d+1)-space respectively. Let Δ be the set of all these cliques and turn Δ into a graph by defining $C \in \Delta(D)$ for $C,D \in \Delta$ iff $|C \cap D| = 1$. Then Δ has two connected components Δ^1, Δ^2 associated with the above partitioning of cliques in two classes. If $d = 2$, the points of Δ^1, Δ^2 correspond to the projective points and planes respectively ot the projective space on V. A classical result of projective geometry [15], yields that $\mathrm{Aut}\,\Delta^1 = \mathrm{Aut}\,\Delta^2 \cong P\Gamma L(n,q)$.

If $n > 2d \geq 4$, then $\mathrm{Aut}(\Delta^1) \cong \mathrm{Aut}(\Delta) \cong P\Gamma L(n,q)$ and we are done. If $n = 2d \geq 4$, any polarity interchanging projective points and hyperplanes yields an additional involution of $\mathrm{Aut}(Ja)$ and hence of $\mathrm{Aut}(\Delta)$ interchanging Δ^1 and Δ^2. It is well known [15] that the group generated by $P\Gamma L(n,q)$ and such a polarity is the full automorphism group of $P\Gamma L(n,q)$. This settles the additional property. □

2.2.5. No characterization of Ja by parameters is known to the author. The following result might serve as a first step in this direction. It is an improved version of Cooperstein's Theorem A in [8]. We recall from [8] that a graph Γ (together with its collection of singular lines) is called a *polar space* if $\gamma^{\perp} \cap L \neq \emptyset$ for any $\gamma \in \Gamma$ and (singular) line L. A polar space Γ is called *nondegenerate* if $\gamma^{\perp} \neq \Gamma$ for any $\gamma \in \Gamma$ and a *generalized quadrangle* if each line is a maximal clique. A *singular subspace* Δ of Γ is a clique Γ such that $|L \cap \Delta| = 0, 1, |L|$ for all singular lines L. Finally, the *rank* of a polar space is the maximal number k such that there exists a chain $\emptyset = \Delta_0 \subsetneq \Delta_1 \subsetneq \dots \subsetneq \Delta_k$ of singular subspaces in Γ.

THEOREM. ([7]). *Let Γ be a connected non-complete garph in which all singular lines have at least three points. Suppose that $\{\gamma,\delta\}^\perp$ is a non-degenerate generalized quadrangle for any $\gamma \in \Gamma$ and $\delta \in \Gamma_2(\gamma)$, and that $|\gamma^\perp \cap L^\perp| \neq 1$ for any $\gamma \in \Gamma$ and singular line L. Then Γ is either a polar space of rank 3 or one of the graphs* Ja.

2.3. Dual polar spaces

2.3.1. DEFINITIONS. In the sequel q, r are prime powers. Let V be one of the following spaces equipped with a form.

$C_d(q) = \mathbb{F}_q^{2d}$ with a nondegenerate symplectic form.

$B_d(q) = \mathbb{F}_q^{2d+1}$ with a nondegenerate quadratic form.

$D_d(q) = \mathbb{F}_q^{2d}$ with a nondegenerate quadratic form of Witt index d.

${}^2D_{d+1}(q) = \mathbb{F}_q^{2d+2}$ with a nondegenerate quadratic form of Witt index d.

${}^2A_{2d}(r) = \mathbb{F}_q^{2d+1}$ with a nondegenerate hermitian form ($q = r^2$).

${}^2A_{2d-1}(r) = \mathbb{F}_q^{2d}$ with a nondegenerate hermitian form ($q = r^2$).

Background on these spaces and their forms can be found in [15]. The spaces $C_d(q)$, $B_d(q)$, $D_d(q)$, ${}^2D_{d+1}(q)$, ${}^2A_{2d}(r)$, ${}^2A_{2d-1}(r)$ are often named $Sp(2d,q)$, $\Omega(2d+1,q)$, $\Omega^+(2d,q)$, $\Omega^-(2d+2,q)$, $U(2d+1,r)$ and $U(2d,r)$ respectively.

A subspace of V is called *isotropic* whenever the form vanishes completely on this subspace. Maximal isotropic subspaces have dimension d (in other words, are d-spaces of V). The *dual polar graph* E (*of diameter* d *on* V) has for vertices the maximal isotropic subspaces. Two points X, Y of E are *adjacent* iff $\dim X \cap Y = d-1$.

Let e be 0, 0, -1, 1, $\frac{1}{2}$, $-\frac{1}{2}$ in the respective cases $C_d(q)$, $B_d(q)$, $D_d(q)$, ${}^2D_{d+1}(q)$, ${}^2A_{2d}(r)$, ${}^2A_{2d-1}(r)$.

2.3.2. LEMMA.

(i) *The number of isotropic* 1-*spaces in* V *is* $\left[{d \atop 1}\right](q^{d+e}+1)$.

(ii) *The number of isotropic* k-*spaces in* V *is* $\left[{d \atop k}\right] \prod_{i=0}^{k-1} (q^{d+e-i}+1)$.

PROOF.

(i) See [1].

(ii) For $k = 1$ the formula is that of (i). Let $k > 1$. The number of isotropic k-spaces of V containing a given isotropic $(k-1)$-space W is $\left[{d-k+1 \atop 1}\right](q^{d+e+1-k}+1)$ according to (i). By induction, there are $\left[{d \atop k-1}\right] \prod_{i=0}^{k-2} (q^{d+e-i}+1)$ such W. As each isotropic k-space contains $\left[{k \atop 1}\right]$ such W, the desired number is

$$\left[{d-k+1 \atop 1}\right](q^{d+e+1-k}+1)\left[{d \atop k-1}\right] \prod_{i=0}^{k-2} (q^{d+e-i}+1)/\left[{k \atop 1}\right].$$

This proves the lemma. □

2.3.3. PROOF OF THEOREM (E). First of all, v results from (2.3.2.ii) upon substitution of $k = d$. Distance in E is as in Ja: $X,Y \in E$ are of distance j iff $\dim (X\cap Y) = d-j$. In view of Witt's theorem [15], E is distance-transitive.

Taking $X,Y \in E$ at distance j, we obtain c_j as the number of $(d-1)$-spaces in Y containing $X \cap Y$, since to any such $(d-1)$-space U corresponds the maximal isotropic space $U + U^{\perp} \cap X$ in $\Gamma_{j-1}(X) \cap \Gamma(Y)$. This yields $c_j = \left[{j \atop j-1}\right] = \left[{j \atop 1}\right]$.

We shall now compute b_j as $b(X,Y)$ for $X,Y \in E$ at distance j. Suppose $Z \in E(X) \cap E_{j+1}(Y)$. Then $\dim X \cap Z = d-1$ and $\dim(Y\cap Z) = d-j-1$, so $\dim(X\cap Y\cap Z) = d-j-1$. On the other hand, if U is a $(d-1)$-space in X such that $U \cap Y$ is a $(d-j-1)$-space (there are $\left[{d \atop 1}\right] - \left[{j \atop 1}\right]$ such spaces U), there are q^{e+1} maximal isotropic spaces Z with $Z \cap X = U$ as $q^{e+1} + 1$ is the number of isotropic 1-spaces of V/U by (2.3.2.i). We claim that any such Z satisfies $\dim(Z \cap Y) = d-j-1$. For, if $z \in Z \cap Y$, then $X \cap Z + X \cap Y + \mathbb{F}_q z$ is an isotropic space containing X, so $z \in X$ by maximality of X. Thus $Z \cap Y = X \cap Z \cap Y = U \cap Y$ is of dimension $d-j-1$ as claimed. It results that $Z \in E(X) \cap E_{j+1}(Y)$ iff $Z \cap X$ is a $(d-1)$-space and $Z \cap X \cap Y$ is a $(d-j-1)$-space and that $b_j = |E(X)\cap E_{j+1}(Y)| = q^{e+1}(\left[{d \atop 1}\right]-\left[{j \atop 1}\right]) = q^{j+e+1}\left[{d-j \atop 1}\right]$. Finally, $\{X,Y\}^{\perp\perp}$ for distinct collinear X,Y consists of the maximal isotropic subspaces containing $X \cap Y$. Therefore $1+s$ is the number of isotropic 1-spaces of $V/X \cap Y$, which is $q^{e+1} + 1$ as we have seen before. Thus $s = q^{e+1}$. □

2.3.4. ADDITIONAL PROPERTY. If $d \geq 2$, then $\mathrm{Aut}(E) \cong P\Sigma p(2d,q)$, $P\Gamma O(2d+1,q)$, $P\Gamma O^{+}(2d,q)$, $P\Gamma O^{-}(2d+2,q)$, $P\Gamma U(2d+1,r)$, $P\Gamma U(2d,r)$ in the respective cases

$E = C_d(q),\ B_d(q),\ D_d(q),\ {}^2D_{d+1}(q),\ {}^2A_{2d}(r),\ {}^2A_{2d-1}(r).$

PROOF. E determines the underlying polar space (cf. [6]) uniquely, so that Aut(E) is the full group of automorphisms of this polar space. The result is therefore a consequence of Theorem 8.6 in [29].

2.3.5. DEFINITION (SHULT & YANUSHKA [24]). A distance-regular graph Γ of diameter d is called a *regular near* 2d-*gon* if the set h of maximal cliques in Γ has the following properties:

(i) Each $L \in h$ has cardinality $a_1 + 2$ (here, a_1 is defined as in (0.2)).

(ii) For any $\gamma \in \Gamma$ and $L \in h$, there is a unique $\zeta \in L$ such that $d(\gamma,\zeta) = \min_{\delta\in L} d(\gamma,\delta)$.

Note that (i) can be rephrased as $h = \{\{\gamma,\delta\}^{\perp\perp} | \gamma \in \Gamma,\ \delta \in \Gamma(j)\}$. The members of h in a regular near 2d-gon are called *lines*. Thus the lines of such a graph are nothing but the singular lines defined in (0.3).

2.3.6. LEMMA. *Let* Γ *be a distance-regular graph (with intersection array* $\{b_0,b_1,\ldots,b_d;c_1,c_2,\ldots,c_d\}$, *as always) such that* s *exists and* $s = a_1 + 2$ *(cf.* 0.3*). Put* $c_0 = 0$. *Then* Γ *is a regular near* 2d-*gon iff* $b_i = s(c_d-c_i)$ *for each* $i \in \{0,1,2,\ldots,d\}$.

PROOF. The existence of s is equivalent to property (i) of a near 2d-gon.

Let Γ be a regular near 2d-gon. Assume first that $\gamma,\delta \in \Gamma$ are of distance d (the diameter of Γ). Then any line through δ must have a unique point of $\Gamma_{d-1}(\gamma)$. Since any point of $\Gamma_{d-1}(\gamma) \cap \Gamma(\delta)$ determines a unique line through δ, there are c_d lines through δ. It follows that there are c_d lines through any point of Γ. Now assume that $\gamma,\delta \in \Gamma$ are of distance j ($1\le j\le d$). Any line through δ either has a unique point in $\Gamma_{j-1}(\gamma)$ or has no points in $\Gamma_{\le j}(\gamma)\setminus\{\delta\}$ at all. There are c_j lines through δ bearing a point in $\Gamma_{j-1}(\gamma)$ (by the same argument as before for $j = d$). The remaining lines, $c_d - c_j$ in number, therefore account for all points in $\Gamma_{j+1}(\gamma) \cap \Gamma(\delta)$. It results that $b_j = s(c_d-c_j)$.

In order to obtain the reverse implication, assume $b_i = s(c_d-c_i)$ for all $i \in \{0,1,2,\ldots,d\}$, and let $\gamma \in \Gamma$, $\delta \in \Gamma_j(\gamma)$. By induction on j we show that there are c_j lines L through δ such that $|L\cap\Gamma_{j-1}(\gamma)| = 1$ and that $L \cap \Gamma_{\le j}(\gamma) = \{\delta\}$ for the remaining lines. Clearly, this establishes (ii) of

2.3.3 and hence the lemma.

For $j = 1$ the line $L = \{\gamma,\delta\}^{\perp\perp}$ is the unique one on δ with $|L \cap \Gamma_0(\gamma)| = 1$. Moreover, $\gamma^{\perp} \cap L = \{\delta\}$ for any other line L on δ by construction of lines.

Let $j > 1$. First, suppose that L is a line on δ such that ζ, η are distinct points of $L \cap \Gamma_{j-1}(\gamma)$. By induction, the fact that L is a line through ζ with $\{\zeta,\eta\} \subseteq \Gamma_{\leq j-2}(\gamma) \cap L$ implies that $L \cap \Gamma_{j-3}(\gamma) \neq \emptyset$. Thus there is $\theta \in \Gamma_{j-3}(\gamma) \cap \Gamma(\delta)$, conflicting $d(\gamma,\delta) = j$.

The conclusion of that the c_j points of $\Gamma_{j-1}(\gamma) \cap \Gamma(\delta)$ correspond to c_j distinct lines L on δ with $L \cap \Gamma_{j-1}(\gamma) \neq \emptyset$. These lines have all together $c_j(s-1)$ points in $\Gamma_j(\gamma) \cap \Gamma(\delta)$. But $a_j = b_0 - b_j - c_j = c_j(s-1)$, so they are all of $\Gamma_j(\gamma) \cap \Gamma(\delta)$. It follows that $L \cap \Gamma_{\leq j}(\gamma) = \{\delta\}$ for any line L on δ with $L \cap \Gamma_{j-1}(\gamma) = \emptyset$. This proves the lemma. □

2.3.7. ADDITIONAL PROPERTY. Let $a_1 \geq 1$ and $d \geq 3$, and suppose Γ is a distance-regular graph of diameter d in which the singular lines have size $a_1 + 2$. If Γ has the same intersection array as E, where E is a dual polar space on ${}^2D_{d+1}(q)$, ${}^2A_{2d}(r)$, ${}^2A_{2d-1}(r)$, then $\Gamma \cong E$. If Γ has the same intersection array as the dual polar space E on $C_d(q)$, then $\Gamma \cong C_d(q)$ or $B_d(q)$.

PROOF. (sketch). In view of 2.3.6, any distance-regular graph Γ with the above mentioned properties is a regular near 2d-gon. Thus the result follows from the CAMERON-SHULT & YANUSHKA Theorem [6] and the classification of polar spaces of rank ≥3 [29], once we have shown that any point is 'classical' with respect to any 'quad' in the terminology of [24]. But this follows from the equalities $c_{i+1} = c_i(c_2-1)+1$ for $i \in \{1,2,\ldots,d-1\}$ as we shall now sketch.

Let Q be a quad and let $\gamma \in \Gamma$ with $d(\gamma,Q) = i$. It is shown that γ is classical with respect to Q by induction on i. If $i = 0$, there is nothing to prove, if $i = 1$ this follows from the (geodesical closure) property of Q that $|\gamma^{\perp} \cap Q| \leq 1$.

Let $i \geq 1$ and assume all points of $\Gamma_i(Q) = \{\delta \in Q \mid d(\delta,Q) = i\}$ are classical (with respect to Q). Denote by $p(\delta)$ for δ a classical point, the unique point in Q nearest δ.

Take $\delta \in \Gamma_i(Q)$. Then there are c_i lines L with $L \cap \Gamma_{i-1}(Q) \neq \emptyset$, namely those with $L \cap \Gamma_{i-1}(p(\delta)) \neq \emptyset$. We claim that δ is on $c_2(c_{i+1}-c_i)$ lines contained in $\Gamma_i(Q)$ and on c_d-c_{i+2} lines L having classical points only for which

$L \cap \Gamma_i(Q) = \{\delta\}$.

To establish the first part of the claim, note that any line L in $\Gamma_i(Q)$ determines a unique line $p(L)$ in Q on $p(\delta)$ and that any line in Q on $p(\delta)$ (there are c_2 such lines) determines $c_{i+1} - c_i$ lines on δ in $\Gamma_i(Q)$.

The second part of the claim follows from the observation that for fixed $\eta \in \Gamma_{i+2}(\delta) \cap Q = \Gamma_2(p(\delta)) \cap Q$, the lines L through δ having classical points only and satisfying $L \cap \Gamma_i(Q) = \{\delta\}$ are exactly those for which $L \cap \Gamma_{i+1}(\eta) = \emptyset$. So far, we have found $c_i + c_2(c_{i+1}-c_i) + (c_d-c_{i+2}) = c_d$ lines through δ, all bearing points classical with respect to Q. Since δ was arbitrarily chosen and Γ is connected, it follows that $\gamma \in \Gamma_{i+1}(Q)$ is classical, too. This finishes the proof. □

2.3.8. REMARK. The proofs above show that the intersection array of a regular near $2d$-gon Γ which is a dual polar space is determined by a_1, c_2 and d. (Recall that $c_{i+1} = (c_2-1)c_i + 1$ and $b_i = (a_1+1)(c_d-c_i)$.) Since $c_2 = q+1$ and $a_1 = q^{e+1} - 1$, this explains the unifying role of e in the above treatment of dual polar space.

2.4. The Hamming graphs

2.4.1. DEFINITION. Take X to be a finite set of cardinality $q \geq 1$. The *Hamming graphs* H (*of diameter* d *on* X) has vertex set $H = \times_{i=1}^{d} X$, the cartesian product of d copies of X. Two points x, y of H are adjacent whenever they differ in precisely one coordinate.

2.4.2. PROOF OF THEOREM. (H): is straightforward. Note that $x \in H_i(y)$ iff x and y differ in precisely i coordinates. □

2.4.3. ADDITIONAL PROPERTY. If $q \neq 4$, then H is the only distance-regular graph (up to isomorphism) whose intersection array is that of H. For $q = 4$, there are precisely $[d/2]$ (isomorphism classes of such graphs other than H).

PROOF. See EGAWA [17]. In the first (and hardest) part of the proof the lines $\{x,y\}^{\perp\perp}$ are shown to have size $s+1 = q$. Then Lemma 2.3.6 can be applied. It remains to show that a regular near $2d$-gon Γ with $s = q-1$ and $c_i = i$ is isomorphic to H. We shall describe how this can be done.

A map $\lambda: \Gamma \to H$ is set up, which is to be interpreted as a labelling and which will eventually turn out to be an isomorphism of graphs. Choose 0 in Γ and define $\lambda(0) = (0,0,\ldots,0)$. Let $L_1, L_2, \ldots, L_n$ be the lines through 0, and label the points of L_i $(1 \le i \le n)$ such that $\lambda(L_i) = \{(a_j)_{1\le j\le n} \mid a_i = 1$ and $a_j = 0$ for $j \ne i\}$. For each point $\gamma \in \Gamma_j(0)$, let $S(\gamma)$ denote $\Gamma(0) \cap \Gamma_{j-1}(\gamma)$ and label γ with $\lambda(\gamma) = \Sigma_{\delta \in S(\gamma)} \lambda(\delta)$.

By induction on $j \ge 2$, it can be shown that

1) For each $\delta \in H_j(0)$ there is $\gamma \in \Gamma_j(0)$ with $\delta = \lambda(\gamma)$.
2) For each $\gamma \in \Gamma_j(0)$ and $\delta \in \Gamma_{j-1}(0)$, the relations $\gamma \in \Gamma(\delta)$ and $S(\gamma) \supseteq S(\delta)$ are equivalent.
3) $\gamma \in \Gamma(\delta)$ iff $\lambda(\gamma) \in H(\lambda(\delta))$ for all $\gamma, \delta \in \Gamma_j(0)$.

This suffices for the proof. □

2.4.4. ADDITIONAL PROPERTY. Let $q = 2$ and let d be even. The graph H' defined on the points of H by $\gamma \in H'(\delta) \iff \gamma \in H_{d-1}(\delta)$, is isomorphic to H.

PROOF. Note that H' is bipartite; its parts consist of the points of even weight and odd weight respectively, where weight is the distance in H to a fixed point of H. Replacing the points of odd weight by their (unique) opposite point, leads to an isomorphism from H' to H. □

2.5. The q-analog of the Hamming graph

2.5.1. DEFINITION. Take $n \ge 0$. Let Ha be the vector space of $d\times(n+d)$-matrices over $\mathbb{F}_q$. The underlying set is turned into a graph by defining $\gamma, \delta \in$ Ha to be *adjacent* whenever $\mathrm{rk}(\gamma-\delta) = 1$, where rk stands for the rank of a matrix. This graph is called the *q-analog of the Hamming graph on* $d \times (n+d)$*-matrices*, and will be denoted Ha, too.

2.5.2. PROOF OF THEROEM (Ha). Clearly, v is the cardinality of $\mathbb{F}_q^{d(n+d)}$, so $v = q^{d(n+d)}$. Translations of the vector space are automorphisms of the graph, as are left (right-) multiplications by invertible $d\times d$-matrices ($(n+d) \times (n+d)$-matrices). Any field automorphism applied to all coefficients of a matrix leads to a graph automorphism. This explains the existence of a subgroup of the form $\mathbb{F}_q^{d(n+d)}(GL(d,q) \times \Gamma L(n+d,q)/\mathbb{F}_q^*)$ of Aut(Ha). Notice that $\gamma \in \mathrm{Ha}_{\le i}(0)$ iff $\mathrm{rk}(\gamma) \le i$. It follows easily that Ha is distance-transitive and of diameter d.

In order to calculate k, b_j, c_j and s, we set $\delta = 0$, the $d\times(n+d)$ matrix with all entries 0. Observe that any matrix of rank 1 can be written as xy^T, where $x \in \mathbb{F}_q^d\setminus\{0\}$ and $y \in \mathbb{F}_q^{n+d}\setminus\{0\}$, and that $xy^T = x_1y_1^T$ for nonzero $x,x_1 \in \mathbb{F}_q^d$ and $y,y_1 \in \mathbb{F}_q^{n+d}$ implies the existence of a nonzero $\lambda \in \mathbb{F}_q$ for which $x = \lambda x_1$ and $y = \lambda^{-1}y_1$.

Thus, k, being the number of $d\times(n+d)$-matrices of rank 1, equals $(q^d-1)(q^{n+d}-1)/(q-1)$.

Next, let γ be the $d\times(n+d)$-matrix of rank j whose first j diagonal entries are 1 and all whose other entries vanish. Necessary and sufficient for $\zeta \in Ha_1(0)$ to satisfy $rk(\zeta+\gamma) = j+1$ is that $\zeta_{\ell,m} \neq 0$ for some $\ell,m > j$. Therefore,

$$\begin{aligned} Ha_1(0)\setminus Ha_{j+1}(\gamma) &= \{\zeta \in Ha_1(0) \mid \zeta_{\ell,m} = 0 \text{ for all } \ell,m > j\} \\ &= \{\zeta \in Ha_1(0) \mid \zeta_{\ell,m} = 0 \text{ for all } m \text{ and all } \ell > j\} \\ &\cup \{\zeta \in Ha_1(0) \mid \zeta_{\ell,m} = 0 \text{ for all } \ell \text{ and all } m > j\}. \end{aligned}$$

Now both constituents of the union in the right hand side can be seen as sets of matrices of rank 1 of given dimensions, and so does their intersection. Applying the above formula for k (with appropriate dimensions) to these three sets as well as to $Ha_1(0)$, we get:

$$\begin{aligned} &(q^d-1)(q^{n+d}-1)/(q-1) - b_j = \\ &= (q^j-1)(q^{n+d}-1)/(q-1) + (q^j-1)(q^d-1)/(q-1) - (q^j-1)^2/(q-1). \end{aligned}$$

This leads to the desired formula for b_j.

In order to determine c_j, let $\zeta \in Ha_1(0)$ have the form $\zeta = xy^T$. For ζ to be in $Ha_{j-1}(\gamma)$, it is necessary that $\zeta_{\ell,m} = 0$ whenever $\ell > j$ or $m > j$. Thus we assume that $x_\ell = y_m = 0$ for all $\ell,m > j$, so that in fact $x,y \in \mathbb{F}_q^j$ (identified with the subspace of $\mathbb{F}_q^d$, and of $\mathbb{F}_q^{n+d}$, on the first j basis vectors). Now such a $\zeta = xy^T$ for nonzero x, y is in $Ha_{j-1}(\gamma)$ iff $\det(xy^T - I_j) = 0$. Given $x \in \mathbb{F}_q^j\setminus\{0\}$, the number of $y \in \mathbb{F}_q^j$ satisfying this (linear) equation is q^{j-1} (the cardinality of a hyperplane of $\mathbb{F}_q^j$). Since

$\zeta \in Ha_1(0) \cap Ha_{j-1}(\gamma)$ determines x uniquely up to a nonzero scalar multiple (and vice versa, as we have just seen), it follows that $c_j = q^{j-1}(q^j-1)/(q-1)$. We finish by computing s. Put $j = 1$, so that $\gamma_{1,1}$ is the only nonzero entry of γ. Then $\{0,\gamma\}^{\perp} = \{\zeta \in Ha \mid \zeta_{ij} = 0 \text{ for } i,j > 1\}$ so $\{0,\gamma\}^{\perp\perp} = \mathbb{F}_q\gamma$ has cardinality q. Thus $s = q-1$. □

2.6. The alternating forms

2.6.1. DEFINITION. Set $V = \mathbb{F}_q^n$ and let Alt stand for the $n(n-1)/2$-dimensional vector space of (bilinear) alternating forms on V, and let $d = [\frac{n}{2}]$. Thus $f \in \text{Alt}$ iff f is a bilinear form on V and $f(x,x) = 0$ for all $x \in V$. The *graph of alternating forms* (*on* V), denoted by Alt too, is defined on the points of Alt by $\gamma \in \text{Alt}(\delta)$ for $\gamma,\delta \in \text{Alt}$ whenever $\text{rk}(\gamma-\delta) = 2$. Here, $\text{rk}(\gamma) = \dim(V/\text{Rad}\,\gamma)$, where $\text{Rad}\,\gamma = \{x \in V \mid \gamma(x,y) = 0 \text{ for all } y \in V\}$. If $\gamma \in \text{Alt}$ and U is a subspace of V, then $\gamma|U$ denotes the form induced on U by γ.

2.6.2. LEMMA. *Let* $\gamma,\delta \in \text{Alt}$.

(i) $\text{Rad}\,\gamma \cap \text{Rad}\,\delta = \text{Rad}\,\gamma \cap \text{Rad}(\gamma-\delta)$.

(ii) $\text{Rad}\,\gamma + \text{Rad}\,\delta = V \Rightarrow \text{Rad}\,\gamma \cap \text{Rad}\,\delta = \text{Rad}(\gamma-\delta)$.

PROOF. Straightforward. □

2.6.3. LEMMA. *Let* $\gamma,\delta \in \text{Alt}$ *and suppose* $\text{rk}(\gamma) = 2j$ *and* $\text{rk}(\delta) = 2$. *Let* W *be a complement of* $\text{Rad}\,\gamma$ *in* V *and write* $U = \text{Rad}(\gamma-\delta) \cap W$. *Then*

(i) $\text{rk}(\gamma-\delta) = 2(j+1) \Leftrightarrow V = \text{Rad}\,\gamma + \text{Rad}\,\delta$.

(ii) $\text{rk}(\gamma-\delta) = 2(j-1) \Leftrightarrow \text{Rad}\,\gamma \subseteq \text{Rad}\,\delta$ *and* $\text{rk}(\gamma|U) = 2$.

Proof.

(i) Suppose $\text{rk}(\gamma-\delta) = 2(j+1)$. Then by (2.6.2i) and the hypotheses on ranks, $n-2(j+1) \le \dim(\text{Rad}\,\gamma \cap \text{Rad}\,\delta) = \dim(\text{Rad}(\gamma-\delta) \cap \text{Rad}\,\delta) \le n-2(j+1)$, whence $\dim(\text{Rad}\,\gamma + \text{Rad}\,\delta) = 2n - \text{rk}\,\gamma - \text{rk}\,\delta - \dim(\text{Rad}\,\gamma \cap \text{Rad}\,\delta)$ $= 2n-2j-2-(n-2j-2) = n$, so that $\text{Rad}\,\gamma + \text{Rad}\,\delta = V$. Conversely, if $V = \text{Rad}\,\gamma + \text{Rad}\,\delta$, then $\text{Rad}(\gamma-\delta) = \text{Rad}\,\gamma \cap \text{Rad}\,\delta$ by (2.6.2ii), so that $\text{Rad}(\gamma-\delta)$ is of dimension

$$\dim(\text{Rad}\,\gamma) + \dim(\text{Rad}\,\delta) - n = n - 2(j+1).$$

This establishes (i).

(ii) Suppose $\mathrm{rank}(\gamma-\delta) = 2j-2$. Then, by Lemma 1(i) and the hypotheses on ranks, $n-2j \le \dim(\mathrm{Rad}(\gamma-\delta) \cap \mathrm{Rad}\,\delta) = \dim(\mathrm{Rad}\,\gamma \cap \mathrm{Rad}\,\delta) \le n-2j$, whence $\mathrm{Rad}\,\gamma \cap \mathrm{Rad}\,\delta = \mathrm{Rad}\,\gamma$ so that $\mathrm{Rad}\,\gamma \subseteq \mathrm{Rad}\,\delta$. Moreover, $2 = \dim(\mathrm{Rad}(\gamma-\delta)/\mathrm{Rad}\,\gamma) = \dim(\mathrm{Rad}(\gamma-\delta) \cap W) = \mathrm{rk}(\gamma|U)$.

Conversely, if $\mathrm{Rad}\,\gamma \subseteq \mathrm{Rad}\,\delta$ and $\mathrm{rk}(\gamma|U) = 2$, we have $\mathrm{Rad}\,\gamma = \mathrm{Rad}\,\gamma \cap \mathrm{Rad}\,\delta = \mathrm{Rad}(\gamma-\delta) \cap \mathrm{Rad}\,\gamma$ so that $\mathrm{Rad}(\gamma-\delta) \supseteq \mathrm{Rad}\,\gamma$. But the latter two subspaces do not coincide as $U \not\subseteq \mathrm{Rad}\,\gamma$. Since $\dim \mathrm{Rad}\,\delta = n-2$, it follows that $\mathrm{rank}(\gamma-\delta) = \dim \mathrm{Rad}\,\gamma + 2 = n-2j+2$, as wanted. This ends the proof of the lemma. □

2.6.4. PROOF OF THEOREM (Alt). First of all, notice that $\gamma \in \mathrm{Alt}_j(\delta)$ for $\gamma,\delta \in \mathrm{Alt}$ iff $\mathrm{rk}(\gamma-\delta) = 2j$. Thus Alt has diameter d. Moreover, the group $\mathbb{F}_q^{n(n-1)/2} \cdot (GL(n,q)/\{\pm 1\})$ can be seen as automorphism group in much the same way as in the previous section: the action of the group $GL(V) = GL(n,q)$ on γ is given by $(g\gamma)(x,y) = \gamma(g^{-1}x, g^{-1}y)$ for $g \in G$ $(x,y \in V)$. As a result, Γ is distance-transitive and we may compute b_j, c_j, s as $b(0,\delta)$, $c(0,\delta)$, $s(0,\delta)$ for a single appropriately chosen $\delta \in \mathrm{Alt}$.

Computation of b_j. Let $\gamma \in \mathrm{Alt}_j(0)$. By (2.6.3ii) any $\delta \in \mathrm{Alt}_1(0) \cap \mathrm{Alt}_{j+1}(\gamma)$ leads to an (n-2)-space $\mathrm{Rad}\,\delta$ of V intersecting $\mathrm{Rad}\,\gamma$ in an (n-2j-2)-space. Conversely, if U is an (n-2)-space of V such that $U \cap \mathrm{Rad}\,\gamma$ is an (n-2j-2)-space, then there are (q-1) forms $\delta \in \mathrm{Alt}_1(0)$ with $U = \mathrm{Rad}\,\delta$. As the number of such (n-2)-spaces is $q^{4j}\left[{n-2j \atop n-2j-2}\right]$ by (2.2.2iii), we get $b_j = q^{4j}(q^{n-2j}-1)(q^{n-2j-1}-1)/(q^2-1)$ as wanted.

Computation of c_j. Let $\gamma \in \mathrm{Alt}_j(0)$, and let W be a complement of $\mathrm{Rad}\,\gamma$ in V. By (2.6.3.ii) each $\delta \in \mathrm{Alt}_1(0) \cap \mathrm{Alt}_{j-1}(0)$ determines a 2-space U of W with $\mathrm{rk}(\gamma|U) = 2$. On the other hand, if U is a 2-space of W with $\mathrm{rk}(\gamma|U) = 2$, then $\delta \in \mathrm{Alt}$ given by $\delta|U = \gamma|U$ and $\mathrm{Rad}\,\delta = \{v \in V \mid \gamma(v,U) = 0\}$ is the unique form in $\mathrm{Alt}_{j-1}(\gamma) \cap \mathrm{Alt}_1(0)$ with $\delta|U = \gamma|U$. Thus c_j is the number of nonisotropic 2-spaces of W with respect to a nondegenerate alternating form.

Since $\dim W = n-2j$, it results from (2.3.2.ii) that this number is $\left[{2j \atop 2}\right] - \left[{j \atop 2}\right] \prod_{i=0}^{1} (q^{j-i}+1) = q^{2j-2}(q^{2j}-1)/(q^2-1)$, whence c_j.

Computation of s. Let γ be the form in Alt given by $\gamma(x,y) = x_1y_2 - x_2y_1$ for $x = (x_i)_{1\le i\le n}$, $y = (y_i)_{1\le i\le n}$ in V. Now $\{0,\gamma\}^\perp = \{\delta \in \mathrm{Alt}_1(0) \mid \dim(\mathrm{Rad}\,\delta \cap \mathrm{Rad}\,\gamma) \ge n-3\}$, so that for any $\delta \in \{0,\gamma\}^\perp \setminus \mathbb{F}_q\gamma$, we have $\delta^\perp \not\subseteq \{0,\gamma\}^\perp$.

Consequently, $\{0,\gamma\}^{\perp\perp} = \mathbb{F}_q\gamma$ (note that $\mathbb{F}_q\gamma \subseteq \{0,\gamma\}^{\perp\perp}$), whence $s = q-1$.

This settles Theorem (Alt). □

2.7. The hermitian forms

2.7.1. DEFINITION. Set $V = \mathbb{F}_q^d$, where $q = r^2$ for r a prime power, and let Her stand for the d^2-dimensional vector space over $\mathbb{F}_r$ of the hermitian forms on V. The *graph of hermitian forms*, also denoted by Her, on these forms is defined by $\gamma \in \mathrm{Her}(\delta)$ for $\gamma,\delta \in \mathrm{Her}$ iff $\mathrm{rk}(\gamma-\delta) = 1$. Here, rk γ and Rad γ are defined as for Alt in (2.6.1).

2.7.2. PROOF OF THEOREM (Her). As all arguments run parallel to those of the previous section, the proof is left to the reader. □

2.8. The quadratic forms

2.8.1. DEFINITIONS. Set $V = \mathbb{F}_q^n$. A *quadratic form on* V (over $\mathbb{F}_q$) is a map $\gamma: V \to \mathbb{F}_q$ such that

$$\gamma(\lambda x) = \lambda^2\gamma(x) \qquad \text{for all } \lambda \in \mathbb{F}_q \text{ and } x \in V$$

and such that $B_\gamma: V \times V \to \mathbb{F}_q$ defined by

$$B_\gamma(x,y) = \gamma(x+y) - \gamma(x) - \gamma(y) \qquad (x,y \in V)$$

is a bilinear form (the *bilinear form associated with* γ). Let Q denote the $n(n+1)/2$-dimensional vector space of all quadratic forms on V. The *radical* of γ, denoted by Rad γ, is defined by $\mathrm{Rad}\,\gamma = \{x \in \mathrm{Rad}\,B_\gamma \mid \gamma(x) = 0\}$, where, of course, Rad B_γ is defined as in 2.6.1. We observe that $\mathrm{Rad}\,B_\gamma = \mathrm{Rad}\,\gamma$ if q is odd and that $\dim(\mathrm{Rad}\,B_\gamma) \leq \dim(\mathrm{Rad}\,\gamma) + 1$ in general (cf. [15]). The *rank* of $\gamma \in Q$, denoted rk γ, is the number $\mathrm{rk}\,\gamma = \dim(V/\mathrm{Rad}\,\gamma)$. The *graph of quadratic forms on* V has vertex set Q; adjacency for $\gamma,\delta \in Q$ is defined by $\mathrm{rk}(\gamma-\delta) \in \{1,2\}$. This graph will be denoted by Q.

In the proof of theorem (Q), we need a partial subgraph Q' of Q whose vertex set coincides with Q. Adjacency for $\gamma,\delta \in Q'$ is defined by $\mathrm{rk}(\gamma-\delta) = 1$. Obviously, Q is determined as the graph obtained from Q' by letting $\gamma,\delta \in Q$ be adjacent iff $\gamma \in Q'_{\le 2}(\delta)\backslash\{\delta\}$.

2.8.2. LEMMA. *Let* $\gamma,\delta,\zeta \in Q$. *Then*

(i) $\mathrm{rk}(\gamma-\delta) + \mathrm{rk}(\delta-\zeta) \ge \mathrm{rk}(\gamma-\zeta)$

(ii) $\mathrm{Rad}\ \gamma \cap \mathrm{Rad}\ \delta = \mathrm{Rad}\ \gamma \cap \mathrm{Rad}(\gamma-\delta)$.

(iii) $\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ B_\delta = \mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ B_{(\gamma-\delta)}$.

(iv) $\mathrm{Rad}\ B_\gamma + \mathrm{Rad}\ B_\delta = V \Rightarrow \mathrm{Rad}\ B_{(\gamma-\delta)} = \mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ B_\delta$.

PROOF. Straightforward. □

2.8.3. LEMMA. *Let* q *be even. If* $\gamma,\delta \in Q$ *satisfy* $\mathrm{rk}(\gamma+\delta) = 2j-2$, $\mathrm{rk}\ \delta = 2$ *and* $\mathrm{rk}\ \gamma = 2j-1$, *then* $\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta = \mathrm{Rad}\ \gamma$.

PROOF. Suppose $x \in \mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta\backslash\mathrm{Rad}\ \gamma$. Then $x \in \mathrm{Rad}\ B_{(\gamma+\delta)}\backslash\mathrm{Rad}(\gamma+\delta)$, hence $\mathrm{rk}(\gamma+\delta)$ is odd. This conflicts with the fact that $\gamma+\delta$ is an alternating form. Thus $\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta \subseteq \mathrm{Rad}\ \gamma$. As $\mathrm{rk}\ \delta$ is even, $\mathrm{rk}\ \delta = \mathrm{rk}\ B_\delta$, so that $\dim(\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta) \in \{n-2j,\ n-2j+1\}$. But if $\dim(\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta) = n-2j$, then $\mathrm{Rad}\ B_\gamma + \mathrm{Rad}\ B_\delta = V$; thus (2.8.2.iv) yields

$$n-2j+2 = \dim \mathrm{Rad}(\gamma+\delta) \le \dim \mathrm{Rad}(B_{(\gamma+\delta)}) =$$

$$= \dim(\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ B_\delta) = n-2j,$$

which is absurd.

The conclusion is that $\dim(\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta) = n-2j+1 = \dim(\mathrm{Rad}\ \gamma)$, whence $\mathrm{Rad}\ B_\gamma \cap \mathrm{Rad}\ \delta = \mathrm{Rad}\ \gamma$. □

2.8.4. LEMMA. *The group* $G = \mathbb{F}_q^{n(n+1)/2} \times (\Gamma L(V)/\{\pm 1\})$ *is a transitive subgroup of* $\mathrm{Aut}(Q')$ *whose stabilizer* $\Gamma L(V)/\{\pm 1\}$ *of the origin* 0 *has precisely two orbits in* $Q'_m(0)$ *for* $0 < m \le n$, *except for* q *is even and* m *is odd, when there is only one orbit. A representative form of each of these orbits is* $\delta_{m,\varepsilon}$ *for* $\varepsilon = +$ *if* q *is even and* m *is odd and* $\varepsilon \in \{-,+\}$ *otherwise where* $\delta_{m,\varepsilon}$ *is given in Table* II.

TABLE II

Representative forms of orbits in $Q'_m(0)$ under G for $m \in \{2j-1,2j\}$

	$\delta_{2j-1,+}$	$\delta_{2j-1,-}$	$\delta_{2j,+}$	$\delta_{2j,-}$
q odd	$\sum_{i=1}^{j-1} x_{2i}x_{2i-1}+x_{2j-1}^2$	$\sum_{i=1}^{j-1} x_{2i}x_{2i-1}+\lambda x_{2j-1}^2$	$\sum_{i=1}^{j} x_{2i}x_{2i-1}$	$\sum_{i=1}^{j} x_{2i}x_{2i-1}+x_{2j-1}^2+\lambda x_{2j}^2$
q even	$\sum_{i=1}^{j-1} x_{2i}x_{2i-1}+x_{2j-1}^2$	——	$\sum_{i=1}^{j} x_{2i}x_{2i-1}$	$\sum_{i=1}^{j-1} x_{2i}x_{2i-1}+\lambda x_{2j-1}^2+x_{2j-1}x_{2j}+\lambda x_{2j}^2$

$$\lambda \in \mathbb{F}_q \text{ satisfies} \begin{cases} x^2-\lambda \text{ irreducible if } q \text{ is odd} \\ \lambda x^2+x+\lambda \text{ irreducible if } q \text{ is even} \end{cases}$$

2.8.5. <u>SKETCH OF PROOF OF THEOREM</u> (Q) (EGAWA [18]). As Aut(Q') is a subgroup of Aut(Q), the group G of 2.8.4 is a group of automorphisms of Q. This case is harder to deal with than the previous ones as Aut(Q) is not distance-transitive (see 2.8.7). At any rate, Aut(Q) is transitive, so Q is regular. Its degree k is $|Q'_1(0)| + |Q'_2(0)|$. Since the number of quadratic forms of rank 2 on a 2-space is $q^2(q-1)$, we get $k = (q^n-1)+[\frac{n}{2}]q^2(q-1) = (q^{n+1}-1)(q^n-1)/(q^2-1)$. Now $\gamma \in Q_j(\delta)$ iff $\mathrm{rk}(\gamma-\delta) \in \{2j,2j-1\}$, in view of (2.8.2.i) so Q has diameter $d = [\frac{n+1}{2}]$.

Moreover, $k = b(0,\delta) + a(0,\delta) + c(0,\delta)$ for $\delta \in Q$, so in order to establish that Q is distance-regular, it suffices to show that $b(0,\delta)$ and $c(0,\delta)$ are independent of the chosen $\delta \in \mathrm{Quad}_m(0)$ for all $m \le d$. In view of 2.8.4, the numbers $b(0,\delta)$ and $c(0,\delta)$ are determined by the values $b(0,\delta_{m,\varepsilon})$ and $c(0,\delta_{m,\varepsilon})$ for $\delta_{m,\varepsilon}$ as in Table II. Thus the fact that Q is distance-regular results from Lemma 2.8.6 below.

Finally, it is easily checked that $\{0,\delta\}^{\perp\perp} = \mathbb{F}_q\delta$ for $\delta \in Q'_1(0)$ and $\{0,\delta\}^{\perp\perp} = \{0,\delta\}$ for $\delta \in Q'_2(0)$, where $\perp$ is taken with respect to Q. Thus $s(\gamma,\delta) \in \{q-1,1\}$ for $\delta \in Q_1(\gamma)$. □

TABLE III

$d_{h,\ell}(\delta_{2j-1,\varepsilon})$	$\ell = 1$	$\ell = 2$
$h = 2j-3$	0	$\frac{q^{2j-2}(q^{2j-4}-1)}{q^2-1}$
$h = 2j-2$	q^{2j-2}	$q^{2j-2}(q^{2j-2}-1)$
$h = 2j+1$	0	$\frac{q^{4j}(q^{n-2j}-1)(q^{n-2j+1}-1)}{q^2-1}$
$h = 2j+2$	0	0

$d_{h,\ell}(\delta_{2j,\varepsilon})$	$\ell = 1$	$\ell = 2$
$h = 2j-3$	0	0
$h = 2j-2$	0	$\frac{q^{2j-2}(q^{2j}-1)}{q^2-1}$
$h = 2j+1$	$q^{2j}(q^{n-2j}-1)$	$q^{2j}(q^{n-2j}-1)(q^{2j}-1)$
$h = 2j+2$	0	$\frac{q^{4j+2}(q^{n-2j}-1)(q^{n-2j-1}-1)}{q^2-1}$

2.8.6. <u>LEMMA</u>. *For* $\ell \in \{1,2\}$, $h \in \{2j-3, 2j-2, 2j+1, 2j+2\}$ *and* $\delta \in \{\delta_{2j-1,\varepsilon}, \delta_{2j,\varepsilon}\}$ *where* $\varepsilon \in \{-,+\}$, *put* $d_{h,\ell}(\delta) = |Q'_\ell(0) \cap Q'_h(\delta)|$.

Then $d_{h,\ell}(\delta)$ is as given in Table III.

PARTIAL PROOF. All zero entries in Table III are explained by (2.8.2.i). Since the proof of this lemma consists of numerous steps, many of which are much alike, we shall restrict to the case where q is even and $\delta = \delta_{2j-1,+}$. Let thus q be even and $\delta = \delta_{2j-1,+}$. Take $e_1, e_2, \dots, e_{2j-1}$ a linearly independent set of vectors in V such that

$$\delta\left(\sum_{i=1}^{2j-1} x_i e_i\right) = \sum_{i=1}^{j-1} x_{2i}x_{2i-1} + x_{2j-1}^2,$$

and let W be the $(2j-2)$-space spanned by $e_1, e_2, \dots, e_{2j-2}$. We shall determine $d_{2j-3,2}(\delta)$ first. Suppose that $\gamma \in Q_2'(0) \cap Q_{2j-3}'(\delta)$. Then $rk(\gamma) = 2$, $rk\, B_{(\gamma+\delta)} = 2j-4$ and $rk\, B_\delta = 2j-2$, so $\mathrm{Rad}\, B_\delta \subseteq \mathrm{Rad}\, B_\gamma$ by (2.6.3.ii). Also, by Lemma (2.6.3.ii), B_δ corresponds to a 2-space U of W which is not isotropic with respect to B_γ; and conversely any such U determines B_γ uniquely by $rk\, B_\gamma = 2$ and $\gamma|U = \delta|U$. Given U, there are q^2 choices for γ such that $rk\,\gamma = 2$ and $B_{\gamma|U} = B_{\delta|U}$. This proves that $d_{2j-3,2}(\delta)$ is q^2 times the number of 2-spaces in a $(2j-2)$-space that are isotropic with respect to $B_{\delta_{2j-2,+}}$. Therefore,

$$\begin{aligned} d_{2j-3,2}(\delta) &= q^2\left(\left[{2j-2 \atop 2}\right] - \left[{j-1 \atop 2}\right](q^{j-1}+1)(q^{j-2}+1)\right) \\ &= q^{2j-2}(q^{2j-4}-1)/(q^2-1) \end{aligned}$$

by (2.3.2.ii), as wanted.

We shall now verify the formula for $d_{2j-2,1}(\delta)$. Suppose $\gamma \in Q_1'(0) \cap Q_{2j-2}'(\delta)$. Then $B_\gamma = 0$, and $\mathrm{Rad}(\gamma+\delta) = \mathrm{Rad}\, B_{(\gamma+\delta)} = \mathrm{Rad}\, B_\delta$, whence $(\gamma+\delta)\mathrm{Rad}\, B_\delta = 0$. Thus $\gamma(e_{2j-1}) = 1$ and there are $\alpha_1, \dots, \alpha_{2j-2} \in \mathbb{F}_q$ such that

$$\gamma\left(\sum_{i=1}^{2j-2} x_i e_i\right) = \sum_{i=1}^{2j-2} \alpha_i x_i^2.$$

Conversely, any $\gamma \in Q$ with $\mathrm{Rad}\, B_\delta \subseteq \mathrm{Rad}\,\gamma$ and

$$\gamma\left(\sum_{i=1}^{2j-1} x_i e_i\right) = \sum_{i=0}^{2j-2} \alpha_i x_i^2 + x_{2j-1}^2$$

is contained in $Q_1'(0) \cap Q_{2j-2}'(\delta)$. Thus $d_{2j-1,1}(\delta)$ is the number of possible choices for $\alpha_1,\dots,\alpha_{2j-2}$ in $\mathbb{F}_q$, whence $d_{2j-1,1}(\delta) = q^{2j-2}$. We continue with $d_{2j-2,2}(\delta)$. Suppose $\gamma \in Q_2'(0) \cap Q_{2j-2}'(\delta)$. Then Rad $\gamma \cap$ Rad B_δ = Rad δ according to 2.8.3. Note that there are

$$q^{2j-3}\begin{bmatrix}2j-2\\2j-3\end{bmatrix} = q^{2j-3}\begin{bmatrix}2j-2\\1\end{bmatrix}$$

(n-2)-spaces U such that U $\cap$ Rad B_δ = Rad δ (cf. (2.2.2.ii)).

We may (and shall) assume that there are $f_0 \in$ Rad $B_\delta\backslash$Rad δ, and a basis $f_1,\dots,f_{2j-2}$ of a complement of Rad B_δ in V such that $f_1,\dots,f_{2j-3}$ are in U and

$$B_\delta\Big(\sum_{i=1}^{j-1} X_i f_i, \sum_{i=1}^{j-1} Y_i f_i\Big) = \sum_{i=1}^{j-1} X_{2i}Y_{2i-1} + X_{2i-1}Y_{2i})$$

Let $x,y,z \in \mathbb{F}_q$ be such that $\gamma(f_{2j-2}) = x$, $\gamma(f_0) = y$ and $B_\gamma(f_{2j-2},f_0) = z$. Then $z \neq 0$ as rk $\gamma = 2$.

Now Rad$(B_{\gamma+\delta}) = (zf_{2j-3} + f_0)$ + Rad δ, so rk$(\gamma+\delta) = 2j-2$ iff $(\gamma+\delta)(zf_{2j-3}+f_0) = 0$. We obtain that $\gamma \in Q_2'(0) \cap Q_{2j-2}'(\delta)$ iff $z^2\delta(f_{2j-3}) + \delta(f_0) + x = 0$. Thus, given U, there are $q(q-1)$ triples (x,y,z) such that $z \neq 0$ and $z^2\delta(f_{2j-3}) + \delta(f_0) + x = 0$. It follows that

$$d_{2j-2,2}(\delta) = q(q-1)\, q^{2j-3}\begin{bmatrix}2j-2\\1\end{bmatrix} = q^{2j-2}(q^{2j-2}-1),$$

as desired.

The final number we shall determine here is $d_{2j+1,2}(\delta)$. For $\gamma \in Q_{2j+1}'(\delta) \cap Q_2'(0)$, we have rk$(\gamma) = 2$, rk$(\gamma+\delta) = 2j$ and dim(Rad $\gamma \cap$ Rad δ) = $n-2j-1$. There are $q^2(q-1)$ quadratic forms γ of rank 2 with Rank $\gamma \cap$ Rad δ a given $(n-2j-1)$-space of Rad δ. Clearly, all of them satisfy rk $(\gamma+0)$ = $2j+1$. As the number of $(n-2)$-spaces U of V such that U $\cap$ Rad δ is an $(n-2j-1)$-space is $q^{4j-2}\begin{bmatrix}n-2j+1\\2\end{bmatrix}$ by (2.2.2.iii), we get

$$d_{2j+1,2}(\delta) = q^2(q-1)\, q^{4j-2}\begin{bmatrix}n-2j+1\\2\end{bmatrix}$$

$$= q^{4j}(q^{n-2j+1}-1)(q^{n-2j}-1)/(q^2-1),$$

and we are done. □

2.8.7. ADDITIONAL PROPERTY. If $n > 3$, then Q is not distance-transitive; hence Q is not isomorphic to Alt (though Q and Alt have identical intersection arrays).

PROOF. (sketch). For $q > 2$, the statement is obvious by $s(0,\delta) = 1,\ q-1$ according as $\delta \in Q_2'(0)$ or $\delta \in Q_1'(0)$. For $q = 2$ a further argument is needed.

If $\gamma,\delta \in Q_1(0)$ denote by $\gamma*\delta$ the unique $\zeta \in Q$ such that

$$\{0,\gamma,\delta\}^{\perp\perp} = \{0,\gamma,\delta,\zeta\} \qquad \text{if } \gamma \in Q_1(\delta)$$

and

$$\{\zeta\} = M \cap M^{\perp} \text{ where } M = Q_2(0) \cap \{\gamma,\delta\}^{\perp} \qquad \text{if } \gamma \in Q_2(\delta).$$

Then for $\gamma \in Q_2'(0)$, there are $\zeta,\eta \in Q_2'(0)$ with $\zeta + \eta \in Q_4'(0)$. It follows that $(\zeta*\gamma)*\eta \neq \zeta*(\gamma*\eta)$. On the other hand, if $\gamma \in Q_1'(0)$, then $(\zeta*\gamma)*\eta = \zeta*(\gamma*\eta)$ for all $\zeta,\eta \in Q_1(0)$. Hence, in the case ($n > 3$ and) $q = 2$, the graph Q is not distance-transitive.

Since Alt is distance-transitive, this proves the statements. □

2.9. The polygons

2.9.1. DEFINITION. The n-gon I is the graph whose points are the numbers $1,2,\ldots,n$ and in which $\gamma I \delta$ iff $|\gamma-\delta| \in \{1,n-1\}$ for any two $\gamma,\delta \in I$.

2.9.2. PROOF OF THEOREM. (I): is left to the reader. The diameter d of I is $[n/2]$. □

2.9.3. ADDITIONAL PROPERTIES.

(i) Aut(I) the dihedral group of order $2n$.

(ii) Any distance-regular graph of diameter d with valency 2 is isomorphic to I with $n \in \{2d,2d+1\}$.

(iii) If n is odd, then I' defined on the points of I with $\gamma I' \delta$ iff $\gamma \in I_2(\delta)$ for $\gamma,\delta \in I'$, is a graph isomorphic to I.

3. CONCLUDING REMARKS AND LOOSE ENDS

3.1 Here are some references to results of 'matrix-techniques' applied to specific distance-regular graphs (known to the author). They are chosen so as to contain many other references themselves:

J, O	: Delsarte [10]; Ja: Delsarte [11], [12];
E	: Stanton [27];
H	: Delsarte [10], Stanton [28]; Ha: Delsarte [13];
Alt, Q	: Delsarte & Goethals [14], Stanton [28];
Her	: Stanton [28].

3.2 If the quoted theorems on characterizations of distance-regular graphs by intersection arrays cover the existing literature on this topic, one of the first open problems arising in the context of 0.4 should be to determine all distance-regular graphs (if need be: whose singular lines have size $a_1 + 2$) whose intersection arrays coincide with one of the graphs Ha, Alt or Her.

3.3 Trivalent graphs. In view of (2.9.3.ii), the question arises whether the distance-regular graphs with valency 3 are known. The answer (BIGGS [5]) is that there are exactly 12 such graphs; the largest diameter occurring among them is 8. According to SMITH [25], there are 15 (non-isomorphic) distance-transitive graphs with valency 4; their diameters are at most 12.

3.4 Suppose Γ is a distance-regular graph of diameter $d \geq 6$ for which there exists a number $i \in \{2,3,\ldots,d\}$ such that a new distance-regular graph Γ' results on the points of Γ from the definition $\gamma \in \Gamma'(\delta)$ whenever $\gamma \in \Gamma_i(\delta)$ for $\gamma,\delta \in \Gamma'$. Then, according to BANNAI and BANNAI [3], we have $i \in \{2,d-2,d\}$. We note that (2.1.5), (2.4.4), (2.9.3.iii) are instances of this phenomenon.

3.5 The dual polar spaces E of 2.5 are associated with classical (Chevalley) groups, while the automorphism groups of the graphs Alt, Her and Q defined by forms in 2.6, 2.7, 2.8 resemble certain parabolic subgroups of these groups. In graph-theoretic terms, this might lead to a connection between Alt, Her and Q on the one hand and $E_i(X)$ for some $X \in E$ and $i \in \{1,2,\ldots,d\}$ on the other.

The following construction (designed by W.M. Kantor) provides the desired insight in the case where the vector space V underlying E has dimension 2d:

Let E be either a dual polar space or the graph Ja of d-spaces in the 2d-dimensional vector space V. Fix $X \in E$ and $Y \in E_d(X)$, and choose bases $x_1,\dots,x_d$ of X and $y_1,\dots,y_d$ of Y. Then $x_1,\dots,x_d, y_1,\dots,y_d$ is a basis of V. To any $Z \in E_d(X)$, associate the d×d-matrix M(Z) determined by

$$Z = \begin{pmatrix} I & M(Z) \\ 0 & I \end{pmatrix} Y$$

on the given basis of V.

3.5.1. If E = Ja, then $Z \mapsto M(Z)$ defines the bijection between $Ja_d(X)$ and Ha, where the latter is the q-analog of Hamming on d×d-matrices. Moreover, $Z_1, Z_2 \in Ja_d(X)$ have distance j in Ja iff $\dim Z_1 \cap Z_2 = d - j$, which is equivalent to $rk(M(Z_1) - M(Z_2)) = j$. The conclusion is that $Ja_d(X)$ is isomorphic to Ha.

3.5.2. Now, let E be a dual polar space associated with the form $(v,w) \mapsto v^T Aw$ on V $(v,w \in V)$, where the vectors and matrices are given with respect to the above basis of V, and

$$A = \begin{pmatrix} 0 & I_d \\ \varepsilon I_d & 0 \end{pmatrix} \qquad \text{for } \varepsilon \in \{1,-1\}.$$

Then Z is isotropic iff $M(Z)^T + \varepsilon M(Z) = 0$. So if V is of type Sp(2d,q) (take $\varepsilon = -1$), then $Z \mapsto M(Z)$ leads to an isomorphism from $E_d(X)$ to the graph Q' of quadratic forms on a d-dimensional vector space as defined in (2.8.1).

Similarly, if V is of type $\Omega^+(2d,q)$ (take $\varepsilon = 1$) and q is odd, then $Z \mapsto M(Z)$ induces a bijection from $E_d(X)$ onto Alt, the alternating forms on a d-dimensional vector space. Note that two points of $E_d(X)$ cannot be adjacent in E and that they have distance 2 in E iff their images in Alt are adjacent.

Finally, let V be of type U(2d,r), where $q = r^2$, and consider the form $(v,w) \mapsto \bar{v}^T Aw$, $(v,w \in V)$ for A as above. Let $\pi \in \mathbb{F}_q$ have norm -1. Then Z is isotropic iff M(Z) is anti-hermitian, so that $Z \mapsto \pi M(Z)$ yields an isomorphism from $E_d(X)$ onto Her, the graph of hermitian forms on a d-dimensional vector space V over $\mathbb{F}_q$.

3.6 If Γ is a distance-regular graph with $c_2 = 1$, then $\{\gamma,\delta\}^{\perp} = \{\gamma,\delta\}^{\perp\perp}$ for any $\gamma \in \Gamma$ and $\delta \in \Gamma(\gamma)$, so singular lines have sizes $s+1 = a_1+2$. If, moreover $c_i = 1$ for all $i \in \{1,2,\ldots,d\}$ where d is the diameter of Γ, then Γ is a Moore geometry in the sense of DAMERELL [9]. It is shown by DAMERELL, FUGLISTER & GEORGIACODIS (cf. [9]) and OTT [22] that no such graphs $\Gamma \not\cong I$ exist for $d \geq 3$. Recently, DAMERELL & GEORGIACODIS [31] (and partly ROOS & VAN ZANTEN [23]), extended this result to the case where $c_i = 1$ for all $i \in \{1,2,\ldots,d-1\}$ and $1 \leq c_d \leq k$.

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